Quantum Measurement Back-Reaction and Induced Topological Phases

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Abstract

It is shown that a topological vector-potential (Berry phase) is induced by the act of measuring angular momentum in a direction defined by a reference particle. This vector potential appears as a consequence of the back-reaction due to the quantum measurement.

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As is well-known, many of the most common observables (position, velocity, angular momentum, etc..), both in classical mechanics as well as in quantum mechanics, are *relative* observables – they always are defined relative to a system of reference [1, 2]. Indeed, we never measure the absolute position of a particle, but the distance in between the particle and some other object; similarly, we never measure the angular momentum of a particle along an absolute axis, but along a direction defined by some other physical objects. Anything can constitute a "reference system", from macroscopic bodies to microscopic particles, but they are always there, even if, for simplicity, we don't always refer to them explicitly. Obviously, measuring a system relative to a frame of reference implies an interaction in between the system and the reference system (via the measuring apparatus), and thus affects both. It is here that the quantum mechanical case differs considerably from the classical case. The uncertainty principle implies that unlike classical mechanics, the quantum mechanical back-reaction can never be negligible.

In this letter we show that, the "strong" nature of the quantum mechanical back-reaction on the reference system, can in particular cases give rise to an effective topological vector-potential. This induced topological effect can be interpreted as a Berry phase, thus leading to a fundamental relation between quantum measurements and the Berry effect.

To begin with, let us consider a measurement of a half-integer spin $\vec{s}$ in the direction defined by a quantum particle of mass M. In other words, we consider the measurement of the observable $\vec{s}_{\hat{n}} = \hat{n} \cdot \vec{s}$ where $\vec{s}$ is the spin and $\hat{n} = \vec{r}/|\vec{r}|$ is the direction of the reference particle as seen in the laboratory frame of reference. Choosing the reference particle to be free (except for the coupling with the spin during the measurement), and the measuring interaction to be von-Neumann-like, the total Hamiltonian is:

$$H = \frac{\vec{P}^2}{2M} + H_s + g(t) q \hat{n} \cdot \vec{s}. \quad (1)$$

Here H_s stands for the Hamiltonian of the spin system. The measurement is described by

the last term: q is a canonical variable of the measuring device and its conjugate, P_q, plays the role of the "pointer"; $g(t)$ is a time dependent coupling constant, which we shall take to satisfy $\int g(t)dt = 1$. For the special case of a constant coupling, $g(t) = 1/T$ for $0 < t < T$ and zero otherwise, the shift of the position of the pointer yields the average relative spin: $P_q(T) - P_q(0) = \frac{1}{T}\int_0^T \hat{n}\cdot\vec{s}dt$. In the limiting case of $T \to 0$, we obtain $g(t) \to \delta(t)$, which corresponds to the ordinary von-Neumann measurement.

Notice that in the limit of a continuous measurement, for which $g(t) = \mathit{constant}$ in a finite time interval, the von-Neumann interaction term in Eq. (1), has the same form as the well known monopole-like example of a Berry phase [3]. Thus a Berry phase is expected upon rotation of the reference system. But while in Berry's case the interaction is put in "by hand" just to study its consequences, in our case the interaction naturally arises whenever the spin is measured.

As we shall see, the appearance of the Berry phase and the associated vector potential, can be easily obtained by transforming to a quantum reference-frame. There the spin observable becomes directly measurable and the back-reaction felt by the reference particle is precisely given by the requisite Berry vector potential.

Let us consider first the 2-dimensional case. The reference axis is given in terms of the unit vector $\hat{n} = \hat{x}\cos\phi + \hat{y}\sin\phi$, and $\hat{n}\cdot\vec{s} = s_x\cos\phi + s_y\sin\phi$. (Here $\hat{x}$ and $\hat{y}$ and ϕ denote the standard coordinate unit vectors and respectively the polar angle in the laboratory frame of reference.) The last term in the Hamiltonian (1) above, can be simplified by transforming to a new set of variables. The unitary transformation:

$$U_{(2d)} = e^{-i\phi(s_z - \frac{1}{2})} \tag{2}$$

yields the relations:

$$p'_\phi = U^\dagger p_\phi U = p_\phi - (s_z - 1/2) \;\; ; \;\; p'_r = p_r \;\; ; \;\; \vec{r}\,' = \vec{r}, \tag{3}$$

$$s'_x = s_x \cos\phi + s_y \sin\phi \quad ; \quad s'_y = s_y \cos\phi - s_x \sin\phi \quad ; \quad s'_z = s_z. \tag{4}$$

The effect of this unitary transformation is to define new spin variables and a new canonical momentum for the reference particle, while the coordinates of the reference particle (defined with respect to the laboratory) remain unchanged. The extra 1/2 factor in (2) is required in order to preserve the single valueness of the wave function, of the combined spin and reference particle system, with respect to the angular coordinate ϕ. (For an integer spin we drop the 1/2).

Expressing the Hamiltonian in terms of the new variables we obtain:

$$H = \frac{1}{2M}\left(\vec{p}\,' + \frac{s'_z - 1/2}{r}\hat{\phi}\right)^2 + H'_s + g(t)qs'_x. \tag{5}$$

In these variables, the measuring device interacts directly with s'_x. The relative spin s'_x is a measurable observable, which commutes with the total angular momentum, $p'_\phi - 1/2$, since $[s'_x, p'_\phi] = 0$. We notice however that in the new variables the reference particle sees the effective vector potential

$$\vec{A}_{(2d)} = \frac{s'_z - 1/2}{r}\hat{\phi}. \tag{6}$$

The latter describes the back-reaction on the reference frame, which here takes the form of a fictitious magnetic fluxon at the origin $\vec{r} = 0$, with a magnetic flux $\Phi = s'_z - 1/2$ in the $\hat{z}$-direction. In the absence of the measurement, ($g(t) = 0$), the s'_z component of the spin is a constant of motion. Thus, the $2s + 1$ components of the wave function in the s'_z representation evolve independently. The vector potential corresponding to the $s'_z = m_s$ component is $\vec{A}_{(2d)} = (m_s - 1/2)\hat{\phi}/r$, i.e. it corresponds to an integer number of quantized fluxons. Since for all the components the vector potential is equivalent to a pure gauge transformation, it causes no observable effect on the reference particle. On the other hand, during the measurement, the interaction with the measuring device causes a rotation of s'_z, which in turn leads to observable effects. The rotation of s'_z and the exact character of

the associated effects depends on the relative strength of the different terms in (5). In the present work we are interested in the limit of "ideal", (i.e. very accurate) measurements. In this limit the interaction hamiltonian dominates all other terms. Indeed, in order for the measurement to be accurate, the initial position of the pointer, $P_q(0)$ must be precisely known, i.e. $\Delta P_q(0) \to 0$. In turn, this implies that the uncertainty in q is very big, $\Delta q \geq 1/\Delta P_q(0) \to \infty$, that is, the typical values of q in the interaction hamiltonian are infinite. As a consequence, in this limiting case the spin components s'_y and s'_z, which are orthogonal to $\hat{n}$, rotate with infinite frequency, and can be averaged to zero. (In the original variables (1) the spin is a "fast" degree of freedom which follows adiabatically the slow motion of the reference particle.)

More exactly, the typical frequency of rotation of the spin components s'_y and s'_z, associated with the interaction hamiltonian is $\omega_s \approx g/\Delta P_q = 1/(T\Delta P_q)$. This is to be compared with the frequency associated with H'_S, the "free" hamiltonian of the spin, and with the frequency associated to the kinetic term. The later one is the more important as it scales at least as $1/T$; indeed, to see the Berry phase one needs to perform an interference experiment with the reference particle during the time of the measurement, i.e. the duration of the interference experiment $T_{exp} \leq T$. When the ratio of the angular frequencies is $\omega_s/\omega_r \approx T_{exp}/(T\Delta P_q) >> 1$ (which is always reached when the precision of the measurement is increased while keeping all other parameters constant) we are in the adiabatic regime.

In the adiabatic regime corresponding to an ideal measurement the effective vector potential seen by the reference system can therefore be obtained by taking the expectation value of $\vec{A}_{(2d)}$ with respect to the spin wave function:

$$\langle \vec{A}_{(2d)} \rangle = \langle \frac{s_z - 1/2}{r} \hat{\phi} \rangle \approx \frac{1/2}{r} \hat{\phi}. \tag{7}$$

This corresponds to a semi-quantized fluxon at the origin $r = 0$, pointing to the $\hat{z}$ direction.

The total phase accumulated in a cyclic motion of the reference particle around the semi-fluxon yields the topological (path independent) phase:

$$\gamma_n = \oint A_{(2d)} dl = n\pi, \tag{8}$$

where n is the winding number.

Note that in the above case the exact values of g and ΔP_q are essentially irrelevant - all that is needed is for them to be such that the adiabatic regime holds. On the other hand, outside the adiabatic regime, the interaction term does not completely dominate the other terms, the exact values of g and ΔP_q become essential, and the consequences of the measurement are much more complicated; this case is outside of our present interest.

Consider now the case, of a free reference particle in 3-dimensions, the appropriate transformation which maps: $s'_x = U^\dagger_{(3d)} s_x U_{(3d)} = \hat{n} \cdot \vec{s}$ is

$$U_{(3d)} = e^{-i(\theta - \pi/2)s_y} e^{-i\phi(s_z - 1/2)}, \tag{9}$$

where θ and ϕ are spherical angles. [4]

The corresponding 3-dimensional vector-potential is in this case

$$A_{(3d)x} = -s_y \frac{\cos\theta\cos\phi}{r} + \Big(s_z \sin\theta + s_x \cos\theta - 1/2\Big)\frac{\sin\phi}{r\sin\theta}, \tag{10}$$

$$A_{(3d)y} = -s_y \frac{\cos\theta\sin\phi}{r} - \Big(s_z \sin\theta + s_x \cos\theta - 1/2\Big)\frac{\cos\phi}{r\sin\theta}, \tag{11}$$

$$A_{(3d)z} = s_y \frac{\sin\theta}{r}. \tag{12}$$

For the case of an integer spin or angular momentum the 1/2 above is omitted. It can be verified that $\vec{A}_{(3d)}$ corresponds to a pure gauge non-Abelian vector potential. The field strength vanishes locally, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = 0$. Thus the force on the reference particle vanishes. Furthermore since the loop integral, $\oint \vec{A}_{(3d)} \cdot d\vec{r}$, gives rise to a trivial flux $2n\pi$, the manifold is simply connected. (This can be seen by noticing that the magnetic

field, $\nabla \times \vec{A}_{(3d)}$, due to the terms proportional to s_y vanishes. The other terms correspond to a fluxon pointing in the $\hat{z}$-direction with total flux $\Phi = s_z \sin\theta + s_x \cos\theta - 1/2$ which is quantized for spin components along the direction $\pi/2 - \theta$.) Thus, as in the 2-d case, in absence of coupling with the measuring device, $\vec{A}_{(3d)}$ is a pure gauge vector potential.

In the adiabatic limit discussed above, during the measurement we have $\langle s_z \rangle \approx \langle s_y \rangle \approx 0$. The effective vector potential seen by the reference particle

$$\langle \vec{A}_{(3d)} \rangle = \Big(s_x \cos\theta - 1/2 \Big) \frac{\hat{\phi}}{r \sin\theta}, \tag{13}$$

is identical to the (asymptotical, $r \to \infty$) non-Abelian 't Hooft - Polyakov monopole [5] in the unitary gauge. The effective magnetic field, $\nabla \times \langle \vec{A}_{(3d)} \rangle$:

$$\langle \vec{B} \rangle = s_x \frac{\vec{r}}{r^3} \tag{14}$$

corresponds to that of a magnetic monopole at the origin, $r = 0$, with a magnetic charge $m = s_x$.

The topological vector potential obtained above, clearly have an observable manifestation. Upon rotation of the reference particle around the $\hat{z}$ axis, the particle accumulates an Aharonov-Bohm phase:

$$\gamma_n = \oint \vec{A}_{(3d)} \cdot d\vec{r} = -n\pi(1 - \cos\theta), \tag{15}$$

which equals half of the solid angle subtended by the path. The latter can be observable by means of a standard interference experiment. We thus conclude that during a continuous measurement the back-reaction on the reference particle takes the form of a topological vector potential, of a semi-fluxon in 2-dimensions and that of a monopole in 3-dimensions.

Our discussion above can also be restated in terms of Berry phases. Viewing the reference particle as a slowly changing environment, and the spin system as a fast system which is driven by a time dependent 'environment', we can use the Born-Oppenheimer procedure

to solve for the spin's eigenstates. Let us assume for simplicity that g_0 is sufficiently large so H_s can be neglected, and that $g(t)$ is roughly constant. Considering for simplicity the 2-dimensional case, the appropriate eigenstate equation therefore reads

$$gq\hat{n}(\phi) \cdot \vec{s}|\psi(\phi)\rangle = E|\psi(\phi)\rangle, \tag{16}$$

where ϕ is here viewed as the external parameter. For simplicity let us consider the case of $s = 1/2$. We obtain:

$$|\psi_\pm(\phi)\rangle = \frac{1}{\sqrt{2}}\Big(e^{-i\phi/2}|\uparrow_z\rangle \pm e^{+i\phi/2}|\downarrow_z\rangle\Big) \otimes |q\rangle. \tag{17}$$

The eigenfunctions $|\psi_\pm\rangle$ are double-valued in the angle ϕ. Thus a cyclic motion in space, which changes ϕ by 2π, induces a sign change. The latter is due to the 'spinorial nature' of fermionic particles, which as is well known, flips sign under a 2π rotation [6]. To obtain the appropriate Berry phase we need to construct single valued solutions of Eq. (17):

$$|\Psi(\phi)\rangle = e^{-i\phi(s_z+1/2)}|\uparrow_x\rangle = \Big(e^{-i\phi}|\uparrow_z\rangle \pm |\downarrow_z\rangle\Big) \otimes |q\rangle. \tag{18}$$

It then follows that the Berry phase [3]:

$$\gamma_{Berry} = i\oint \langle\Psi(\phi)|\frac{\partial_\phi}{r}|\Psi(\phi)\rangle d\phi = \gamma_n, \tag{19}$$

is identical to the phase, (8), which is induced by the effective semi-fluxon. Similarly, the Berry phase in the 3-dimensional case corresponds to half of the solid angle subtended by the path of the reference particle.

In conclusion, we have shown that the quantum mechanical back-reaction during a measurement induces in certain cases a topological vector potential. The Berry phase can be viewed in this framework, as a necessary consequence of the "strong" nature of the quantum back-reaction.

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